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Resolution of a System of Coupled Schrödinger Equations

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ABSTRACT

In order to solve a system of coupled Schrödinger equations, four analytical variants are given, and their range of applicability is discussed. One of these variants is used in the determination of the Channel Coupling radial wave functions, the Optical Model wave functions and the regular Coulomb function.

1. INTRODUCTION

The central part of the many problems of quantum mechanics is the solution of a system of coupled Schrödinger equations (e.g. Channel Coupling Theory of Nuclear Reactions [1], Theory of Isobaric Analogue Resonances [2], etc.). Usually this system is solved by numerical approximation methods (a Runge–Kutta method, for example). A numerical resolution of a system of N equations requires N numerical integrations of the system with N different initial conditions (the complete solution being a linear combination of these N numerical integrations), so that the difficulties on numerical evaluation of the solution increase with the number of the equations [3].

Instead of a numerical resolution of the system it is preferable to use a power series method (exactly as in theory of some special functions) because the numerical evaluation of the solution and its derivative is reduced to calculating some recurrence relations. This method is faster than a Runge–Kutta method [4]. On the other hand this method gives analytical expresions for the solution and its derivative and, probably, it permits the comparative study of different approximations of the Channel Coupling Theory, as the Optical Model for the elastic scattering and D.W.B.A. for the inelastic scattering.

Recently, J. Chen proposed [5] a power series method in order to solve a system of coupled Schrödinger equations. This method relies on the Gantmacher's method [6] for solving a first order matrix equation. But in Chen's variant the mathematical conditions imposed on the potential make it inapplicable in many cases (e.g. for a nonspherical potential in Channel Coupling Theory). A generalization is presented in this article which leads to four analytical variants as the solutions of coupled Schrödinger equations. One of the variants (IV) coincides with that proposed in the work [5]. In another variant (III) the mathematical conditions imposed on the potential is less restrictive than in Chen's variant so that it becomes applicable in all cases of Channel Coupling Theory, for example. As examples of applications of this variant we deduce Channel Coupling radial wave functions, Optical Model wave functions and regular Coulomb functions [7].

2. The Solving of the System of Coupled Equations

We consider a system of equations of the form

$$d^2X_i/dx^2 = \sum_{k=1}^{n} V_{ik}X_k = V_{ii}X_i + \sum_{k\neq i} V_{ik}X_k \tag{1}$$

V_{ii} has a singularity at origin

$$V_{ii} = \gamma_i(\gamma_i + 1)/x^2 + \beta_i/x + V_i\,.$$

In matrix notation

$$X \equiv \| X_i \|_1^n \qquad V \equiv \| V_{ik} \|_n^n$$

the system is written

$$d^2X/dx^2 = V \cdot X.$$

By the transformation

$$Y \equiv \left| \begin{matrix} dX/dx \\ X \end{matrix} \right|_1^{2n}, \qquad A \equiv \left| \begin{matrix} 0 & V \\ I & 0 \end{matrix} \right|_{2n}^{2n}, \qquad I \equiv \| \delta_{ik} \|_n^n$$

the system takes the form

$$dY/dx = A \cdot Y. \tag{2}$$

Since, at the origin $X(0) = 0$ and $(dX/dx)_{x=0} = C$ we find $Y(0) = \left| \begin{smallmatrix} C \\ 0 \end{smallmatrix} \right|$. Making the nonsingular transformation $B(x)$

$$\psi = B \cdot Y \tag{3}$$

so that in the matrix equation for ψ to have only one regular singularity for $x = 0$

$$d\psi/dx = P^* \cdot Y. \tag{4}$$

It gives

$$P^*B = dB/dx + B \cdot A. \tag{5}$$

This relation is satisfied by three matrices P^*, B and A. Since P^* has not yet been fixed, B is still arbitrary

$$B \equiv \begin{vmatrix} 0 & B_{12} \\ B_{21} & B_{22} \end{vmatrix} \tag{6}$$

with

$$B_{12} = B_{21} \equiv \|(b_i x^{m_i})\,\delta_{ik}\| \qquad (b_i = \text{constant})$$

$$B_{22}B_{12}^{-1} \equiv C \equiv \left\|\left(\sum_j c_{ij}x^j\right)\delta_{ik}\right\| \qquad (c_{ij} = \text{constant})$$

The inverse of the matrix B is

$$B^{-1} = \begin{vmatrix} B_{11} & B_{21}^{-1} \\ B_{12}^{-1} & 0 \end{vmatrix}$$

where (7)

$$B_{11} = -B_{21}^{-1}B_{22}B_{12}^{-1} = -B_{12}^{-1}\cdot C$$

The quantities b_i, c_{ij}, m_i and j are chosen so that P^* should have a singularity of the form $1/x$

$$P^* = P_{-1}/x + P$$

$$P_{-1} \equiv \begin{vmatrix} P_1 & 0 \\ 0 & P_2 \end{vmatrix} \qquad P \equiv \begin{vmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{vmatrix} \tag{8}$$

The different choices lead to four variants

I. $b_i = 1;\ j = 0, -1;\ c_{i0} = \beta_i/2\gamma_i;\quad m_i = 0;\quad c_{i-1} = \gamma_i$

II. $b_i = 1;\ j = 0, -1;\ c_{i0} = \beta_i/2\gamma_i;\quad m_i = \gamma_i;\quad c_{i-1} = \gamma_i$

III. $b_i = 1;\ j = 0, -1;\ c_{i0} = -\beta_i/2(\gamma_i + 1);\ m_i = 0;\quad c_{i-1} = -(\gamma_i + 1)$

IV. $b_i = 1;\ j = 0, -1;\ c_{i0} = -\beta_i/2(\gamma_i + 1);\ m_i = -(\gamma_i + 1);\ c_{i-1} = -(\gamma_i + 1)$

It results

I.
$$\begin{aligned}
B_{12} &= B_{21} = I \\
B_{22} &= \|(\beta_i/2\gamma_i + \gamma_i/x)\,\delta_{ik}\| \\
B_{11} &= -B_{22} \\
P_1 &= -\|\gamma_i\delta_{ik}\| \qquad (\gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_n) \\
P_2 &= -P_1 \\
P_{11} &= -\|(\beta_i/2\gamma_i)\,\delta_{ik}\| \\
P_{12} &= I \\
(P_{21})_{ii} &= V_i - \beta_i^2/4\gamma_i^2 \\
(P_{21})_{ij} &= V_{ij} \qquad (i \neq j) \\
P_{22} &= -P_{11}
\end{aligned}$$

II. $$\begin{aligned}
B_{12} &= B_{21} = \|(x^{\gamma_i})\,\delta_{ik}\| \\
B_{22} &= \|((\beta_i/2\gamma_i)\,x^{\gamma_i} + \gamma_i x^{\gamma_i-1})\,\delta_{ik}\| \\
B_{11} &= -\,\|((\beta_i/2\gamma_i)\,x^{-\gamma_i} + \gamma_i x^{-\gamma_i-1})\,\delta_{ik}\| \\
P_1 &= 0 \\
P_2 &= \|(2\gamma_i)\,\delta_{ik}\| \qquad (\gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_n) \\
P_{11} &= -\,\|(\beta_i/2\gamma_i)\,\delta_{ik}\| \\
P_{12} &= I \\
(P_{21})_{ii} &= V_i - \beta_i^2/4\gamma_i^2 \\
(P_{21})_{ij} &= V_{ij} x^{\gamma_i-\gamma_j} \qquad (i \neq j) \\
P_{22} &= -P_{11}
\end{aligned}$$

III. $$\begin{aligned}
B_{12} &= B_{21} = I \\
B_{22} &= -\,\|(\beta_i/2(\gamma_i+1) + (\gamma_i+1)/x)\,\delta_{ik}\| \\
B_{11} &= -B_{22} \\
P_1 &= \|(\gamma_i+1)\,\delta_{ik}\| \qquad (\gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_n) \\
P_2 &= -P_1 \\
P_{11} &= \|(\beta_i/2(\gamma_i+1))\,\delta_{ik}\| \\
P_{12} &= I \\
(P_{21})_{ii} &= V_i - \beta_i^2/4(\gamma_i+1)^2 \\
(P_{21})_{ij} &= V_{ij} \qquad (i \neq j) \\
P_{22} &= -P_{11}
\end{aligned}$$

IV. $$\begin{aligned}
B_{12} &= B_{21} = \|(x^{-(\gamma_i+1)})\,\delta_{ik}\| \\
B_{22} &= -\,\|((\beta_i/2(\gamma_i+1))\,x^{-(\gamma_i+1)} + (\gamma_i+1)\,x^{-(\gamma_i+2)})\,\delta_{ik}\| \\
B_{11} &= \|((\beta_i/2(\gamma_i+1))\,x^{\gamma_i+1} + (\gamma_i+1)\,x^{\gamma_i})\,\delta_{ik}\| \\
P_1 &= 0 \\
P_2 &= -\,\|\,2(\gamma_i+1)\,\delta_{ik}\| \qquad (\gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_n) \\
P_{11} &= \|(\beta_i/2(\gamma_i+1))\,\delta_{ik}\| \\
P_{12} &= I \\
(P_{21})_{ii} &= V_i - \beta_i^2/4(\gamma_i+1)^2 \\
(P_{21})_{ij} &= V_{ij} x^{\gamma_j-\gamma_i} \qquad (i \neq j) \\
P_{22} &= -P_{11}
\end{aligned}$$

This variant coincides with that from the work [5]. The first two variants require $\gamma_i \neq 0$ and the last two $\gamma_i \neq -1$. If $\gamma_i = 0$ or $\gamma_i = -1$ then $\gamma_i(\gamma_i + 1)/x^2$ disappears. Then, the equation (4) becomes

$$d\psi/dx = (P_{-1}/x + P)\psi \tag{9}$$

In order to solve this equation by the method of Gantmacher it is necessary to make the transformation $P_{-1} \xrightarrow{S} \Lambda$ so that

$$d\psi/dx = (P_{-1}/x + P)\psi \to d\Phi/dx = (\Lambda/x + Q)\Phi \tag{10}$$

and the characteristic values of Λ should satisfy

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant \lambda_{n+1} \geqslant \cdots \geqslant \lambda_{2n}\,. \tag{11}$$

(In the four variants we have accepted such an order of the equations in the system so that $\gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_n$).

By the S transformation

$$\psi \to \Phi = S\psi$$
$$P_{-1} \to \Lambda = SP_{-1}S^{-1}$$
$$P \to \Pi = SPS^{-1}$$

$$\Lambda = \begin{vmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{vmatrix}, \qquad \Lambda_1 = \begin{vmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{vmatrix}, \qquad \Lambda_2 = \begin{vmatrix} \lambda_{n+1} & & 0 \\ & \ddots & \\ 0 & & \lambda_{2n} \end{vmatrix}$$

we find, in the four variants

I.

$$S = \begin{vmatrix} 0 & T \\ I & 0 \end{vmatrix}, \qquad S^{-1} = \begin{vmatrix} 0 & I \\ T & 0 \end{vmatrix}, \qquad T \equiv \begin{vmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{vmatrix}_n^n$$

$$\Lambda_1 = \begin{vmatrix} \gamma_n & & 0 \\ & \ddots & \\ 0 & & \gamma_1 \end{vmatrix}, \qquad \Lambda_2 = \begin{vmatrix} -\gamma_1 & & 0 \\ & \ddots & \\ 0 & & -\gamma_n \end{vmatrix}, \qquad \Pi = \begin{vmatrix} -TP_{11} & TP_{21}T \\ I & P_{11}T \end{vmatrix}$$

II.

$$S = \begin{vmatrix} 0 & I \\ I & 0 \end{vmatrix}, \qquad S^{-1} = \begin{vmatrix} 0 & I \\ I & 0 \end{vmatrix}, \qquad S^{-1} = S,$$

$$\Lambda_1 = \begin{vmatrix} \gamma_n & & 0 \\ & \ddots & \\ 0 & & \gamma_1 \end{vmatrix}, \qquad \Lambda_2 = 0, \qquad \Pi = \begin{vmatrix} -P_{11} & P_{21} \\ I & P_{11} \end{vmatrix}$$

III.

$$S = \begin{vmatrix} T & 0 \\ 0 & I \end{vmatrix}, \qquad S^{-1} = \begin{vmatrix} T^{-1} & 0 \\ 0 & I \end{vmatrix}, \qquad T^{-1} = T,$$

$$\Lambda_1 = \begin{vmatrix} \gamma_n + 1 & & 0 \\ & \ddots & \\ 0 & & \gamma_1 + 1 \end{vmatrix}, \qquad \Lambda_2 = \begin{vmatrix} -(\gamma_1 + 1) & & \\ & \ddots & \\ & & -(\gamma_n + 1) \end{vmatrix},$$

$$\Pi = \begin{vmatrix} TP_{11}T & T \\ P_{21}T & -P_{11} \end{vmatrix}$$

IV.

$$S = \begin{vmatrix} I & 0 \\ 0 & I \end{vmatrix}, \qquad S^{-1} = S,$$

$$\Lambda_1 = 0, \quad \Lambda_2 = \begin{vmatrix} -2(\gamma_1+1) & & 0 \\ & \ddots & \\ 0 & & -2(\gamma_n+1) \end{vmatrix}, \qquad \Pi = P = \begin{vmatrix} P_{11} & I \\ P_{21} & -P_{11} \end{vmatrix}$$

Now considering the condition (11), the system (10) is solved by the Gantmacher method [6]. It is assumed that Π could be expanded into a power series

$$\Pi = \sum_{t=0}^{\infty} \Pi_t x^t$$

This means that

$$P = \sum_{t=0}^{\infty} P_t x^t$$

In all our variants, P_{11}, P_{12} and P_{22} satisfy this condition. In order to realise the same condition in the case of P_{21} matrix it is necessary that

$$V_i = \sum_{t=0}^{\infty} V_i^t x^t$$

and

(a) $$V_{ij} = \sum_{t=0}^{\infty} V_{ij}^t x^t$$

in the cases I and III, and

(b) $$V_{ij} = \sum_{t=\max|\gamma i-\gamma j|}^{\infty} V_{ij}^t x^t$$

in the cases II and IV.

Usually the potentials V_{ij} used in the Channel Coupling formalism [1, 2] satisfy (a) but sometimes not (b). This makes Chen's variant inapplicable in many cases of coupled Schrödinger equations.

With

$$\Phi(x) = G(x)\,\Gamma(x) \tag{12}$$

when

$$G(x) = I + G_1 x + G_2 x^2 + \cdots = \text{a convergent series} \tag{13}$$

the equation (10) becomes

$$d\Gamma/dx = (\Lambda/x + Q) \cdot \Gamma \tag{14}$$

$$Q(x) = \sum_{t=0}^{\infty} Q_t x^t \tag{15}$$

G and Q can be chosen arbitrarily, subject to the constraints (13) and (15) and the relation (16).

$$tG_t + [G_t, \Lambda] = \sum_{s=0}^{t-1} [\Pi_s G_{t-s-1} - G_{t-s-1} Q_s] \tag{16}$$

It is convenient to write the G_t, Π_t, Q_t and Λ in the form

$$G_t \equiv \| g_{ij}^{(t)} \| \qquad \Pi_t \equiv \| \Pi_{ij}^{(t)} \| \qquad Q_t \equiv \| q_{ij}^{(t)} \| \qquad \Lambda \equiv \| \lambda_i \delta_{ij} \| \tag{17}$$

so that, (16) becomes

$$g_{ij}^{(t)}[t - (\lambda_i - \lambda_j)] = \sum_{s=0}^{t-1} \{\Pi_{ik}^{(s)} g_{kj}^{(t-s-1)} - g_{ik}^{(t-s-1)} q_{kj}^{(s)}\} \tag{18}$$

If $t = \lambda_i - \lambda_j$, $g_{ij}^{(t)}$ is chosen 0 and we obtain

$$q_{ij}^{(t-1)} = \Pi_{ij}^{(t-1)} + f_{ij}^{(t)}(\Pi, g, q)$$

If $t \neq \lambda_i - \lambda_j$, $q_{ij}^{(t-1)}$ is chosen 0 and it gives

$$g_{ij}^{(t)} = [\Pi_{ij}^{(t-1)} + f_{ij}^{(t)}(\Pi, g, q)]/[t - (\lambda_i - \lambda_j)]$$

where

$$f_{ij}^{(t)}(\Pi, g, q) \equiv \sum_{s=0}^{t-2} \{\Pi_{ik}^{(s)} g_{kj}^{(t-s-1)} - g_{ik}^{(t-s-1)} q_{kj}^{(s)}\}.$$

By replacing t by $t + 1$ these equations can be iterated to definie $g^{(t)}$ and $q^{(t)}$ for all values of t. Then, Q takes the form

$$Q(x) = \begin{vmatrix} 0 & \cdot & \| q_{ij}^{(m_i-m_j-1)} x^{m_i-m_j-1} \| \\ & \cdot & \\ 0 & & 0 \end{vmatrix}$$

where $m_i \equiv$ whole numbers $\lambda_i (\lambda_i \equiv m_i + \sigma_i, 0 < \sigma_i < 1)$. With $Q(1) = U$, we find

$$Q(x) = x^M (U/x) x^{-M}$$

where

$$M \equiv \| m_i \delta_{ij} \|, \qquad D \equiv \| \sigma_i \delta_{ij} \|, \qquad \Lambda = M + D.$$

Then the system (14) becomes

$$d\Gamma/dx = [\Lambda/x + x^M(U/x)\, x^{-M}] \cdot \Gamma \tag{19}$$

Since the multiplicative Volterra derivative is defined [6] by

$$D_x \Gamma \equiv (d\Gamma/dx) \cdot \Gamma^{-1}$$

(19) takes the form

$$D_x \Gamma = \Lambda/x + x^M(U/x)\, x^{-M}$$

In fact, one can show that

$$\Lambda/x + x^M(U/x)\, x^{-M} = D_x(x^M x^{U+\Lambda-M})$$

and then following solution results

$$\Gamma = x^M x^{U+\Lambda-M} \cdot C', \qquad C' \equiv \begin{vmatrix} C_1 \\ C_2 \end{vmatrix} = \text{constant matrix} \tag{20}$$

This implies that the solution of the system is

$$Y = B^{-1} S^{-1} G \Gamma = B^{-1} S^{-1} G x^M x^{U+\Lambda-M} \cdot C' \tag{21}$$

If the condition $Y(0) = \left| {C \atop 0} \right|$ is imposed we find for the four cases

I.

$$Y = \begin{vmatrix} dX/dx \\ X \end{vmatrix} = \begin{vmatrix} (B_{12}^{-1} T G_{11} + B_{11} G_{21})\, x^{M_1} x^{U_{11}+D_1} \cdot C_1 \\ B_{12}^{-1} G_{21} x^{M_1} x^{U_{11}+D_1} \cdot C_1 \end{vmatrix}$$

II.

$$Y = \begin{vmatrix} dX/dx \\ X \end{vmatrix} = \begin{vmatrix} (B_{12}^{-1} G_{11} + B_{11} G_{21})\, x^{M_1} x^{U_{11}+D_1} \cdot C_1 \\ B_{12}^{-1} G_{21} x^{M_1} x^{U_{11}+D_1} \cdot C_1 \end{vmatrix}$$

III.

$$Y = \begin{vmatrix} dX/dx \\ X \end{vmatrix} = \begin{vmatrix} (B_{11} T G_{11} + B_{12}^{-1} G_{21})\, x^{M_1} x^{U_{11}+D_1} \cdot C_1 \\ B_{12}^{-1} T G_{11} x^{M_1} x^{U_{11}+D_1} \cdot C_1 \end{vmatrix}$$

IV.

$$Y = \begin{vmatrix} dX/dx \\ X \end{vmatrix} = \begin{vmatrix} (B_{11} G_{11} + B_{12}^{-1} G_{21})\, x^{M_1} x^{U_{11}+D_1} \cdot C_1 \\ B_{12}^{-1} G_{11} x^{M_1} x^{U_{11}+D_1} \cdot C_1 \end{vmatrix}$$

Considering that Π_{11} = constant diagonal matrix

$$\Pi_{11} = \|(\Pi_{11}^0)_{ij}\, \delta_{ij}\|$$

one can show that in the cases III and IV, $U_{11} = 0$. It gives

$$f_{ij}^{(t)} = (\Pi_{11}^{0})_{ii}\, g_{ij}^{(t-1)} - \sum_{s=0}^{t-2} g_{ik}^{(t-s-1)} \cdot q_{kj}^{(s)}$$

In this form we can demonstrate that $(q_{11}^{(t)})_{ij} = 0$. Indeed, if $(q_{11}^{(s)})_{ij} = 0 (s = 0, 1, \ldots, t-1)$ we get $g_{ij}^{(t)} = f_{ij}^{(t-1)}/(t - \lambda_i + \lambda_j) = 0$ and therefore $f_{ij}^{(t+1)} = 0$ which implies $q_{ij}^{(t)} = 0$ and $g_{ij}^{(t+1)} = 0$, and so on. Thus, the result is $U_{11} = 0$. Since $U_{11} = 0$, the variants III and IV become

III.

$$Y = \begin{vmatrix} dX/dx \\ X \end{vmatrix} = \begin{vmatrix} (B_{11}TG_{11} + B_{12}^{-1}G_{21})\, x^{A_1} \cdot C_1 \\ B_{12}^{-1}TG_{11}x^{A_1} \cdot C_1 \end{vmatrix}$$

IV.

$$Y = \begin{vmatrix} dX/dx \\ X \end{vmatrix} = \begin{vmatrix} (B_{11}G_{11} + B_{12}^{-1}G_{21})\, C_1 \\ B_{12}^{-1}G_{11} \cdot C_1 \end{vmatrix}$$

because in these cases $x^{A_1} = x^0 = I$. Thus, we obtain the result of the work [5] as the variant IV.

3. Examples

As examples of application of this method we deduce Channel Coupling radial wave functions, Optical Model radial wave functions, regular Coulomb functions and their derivatives.

a. Channel Coupling Radial Wave Functions

In order to obtain Channel Coupling radial wave functions, we use the variant III. The only nonevaluated quantities in this expression are G_{11} and G_{21}. These matrices could be obtained from the recurrence relation (16). With $Q_{11} = Q_{21} = 0$, the recurrence relation takes the form

$$\begin{vmatrix} (t \cdot I - A_1)\, G_{11}^t + G_{11}^t \cdot A_1 \\ (t \cdot I + TA_1T)\, G_{21}^t + G_{21}^t A_1 \end{vmatrix} = \begin{vmatrix} TP_{11}TG_{11}^{t-1} + TG_{21}^{t-1} \\ \sum_{s=0}^{t-1} P_{21}^s TG_{11}^{t-s-1} - P_{11}G_{21}^{t-1} \end{vmatrix}$$

and the solution becomes

$$\begin{vmatrix} dX/dx \\ X \end{vmatrix} = \begin{vmatrix} (B_{11}TG_{11} + G_{21})\, x^{A_1} \cdot C_1 \\ TG_{11}x^{A_1} \cdot C_1 \end{vmatrix}$$

b. Optical Model Radial Wave Functions

In the case of the Optical Model, the system of coupled equations is reduced to one equation, the Optical Model equation. Adequately all matrices are reduced to one element only. In this case $T \equiv 1$ and the variants III and IV give the same solution. Indeed

$$(B_{11}x^{A_1})_{\text{III var}} = (B_{11})_{\text{IV var}}$$

$$(B_{12}^{-1}x^{A_1})_{\text{III var}} = (B_{12}^{-1})_{\text{IV var}}$$

and consequently in the notation of the variant III, the solution is written

$$Y = \begin{vmatrix} (B_{11}G_{11} + G_{21})\, x^{A_1} \cdot C_1 \\ G_{11}x^{A_1} \cdot C_1 \end{vmatrix}$$

The recurrence relation is now more simplified

$$tG_{11}^{t} = P_{11}G_{11}^{t-1} + G_{21}^{t-1}$$

$$(t + 2A_1)\, G_{21}^{t} = \sum_{s=0}^{t-2} P_{21}^{s}G_{11}^{t-s-1} - P_{11}G_{21}^{t-1}$$

or

$$G_{11}^{n+1} = \left[\beta G_{11}^{n} + \sum_{s=0}^{n-1} V_s G_{11}^{n-s-1}\right] \Big/ (n + 1) \cdot [n + 2(\gamma + 1)]$$

$$G_{21}^{n} = (n + 1)\, G_{11}^{n+1} - [\beta/2(\gamma + 1)] \cdot G_{11}^{n}$$

c. Regular Coulomb Function

When $V_0 = -1$, $V_{s\neq 0} = 0$ the Optical Model equation passes into the equation for the motion in the Coulomb field, and consequently the solution of the Optical Model equation becomes the regular Coulomb function. The above relations take the form

$$G_{11}^{n+1} = [\beta G_{11}^{n} - G_{11}^{n-1}]/(n + 1)[n + 2(\gamma + 1)]$$

The recurrence relation is identical with the formula (15) from the work [7], and regular Coulomb function can be written

$$F_\gamma = Cx^{\gamma+1} \cdot G_{11} = Cx^{\gamma+1} \cdot \sum_{n=0}^{\infty} G_{11}^{n}x^{n}$$

The derivative of the regular Coulomb function is

$$F'_\gamma = Cx^\gamma \sum_{n=0}^{\infty} (n + \gamma + 1)\, G_{11}^n x^n$$

These relations are the formulas (22) and (23) in the work [7].

The Coulomb functions are obtained usually (e.g., the work [7]) by very complicated procedures. The advantage of the above method consists in its simplicity.

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